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# On The Maximum Jump Number $M(2k-1,k)$

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**Abstract.** If $n$ and $k$ $(n \geq k)$ are large enough , it is quite difficult to give the value of $M(n,k)$. R. A. Brualdi and H. C. Jung gave a table about the value of $M(n,k)$ for $1 \leq k \leq n \leq 10$. In this paper, we show that $4(k-1) - \lceil\sqrt{k-1}\rceil \leq M(2k-1,k) \leq 4k-7$ holds for $k \geq 6$. Hence, $M(2k-1,k) = 4k-7$ holds for $6 \leq k \leq 10$, which verifies that their conjecture $M(2k+1,k+1) = 4k - \lceil\sqrt{k}\rceil$ holds for $5 \leq k \leq 9$, and disprove their conjecture $M(n,k) < M(n+l_1, k+l_2)$ for $l_1 = 1$, $l_2 = 1$.

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## Introduction and Lemmas

Let $P$ be a finite poset(partially ordered set) and its cardinality $|P| = n$. Let $\mathbf{n}_{\leq}$ denote the $n$-element poset formed by the set $\{1, 2, \ldots, n\}$ with its usual order. Then an order-preserving bijective map $L$: $P \longrightarrow \mathbf{n}_{\leq}$ is called a linear extension of $P$ to a totally ordered set. If $P = \{x_i \mid 1 \leq i \leq n\}$, then we can simply express a linear extension $L$ by $x_1 - -x_2 - - \cdots - -x_n$ with the property $x_i < x_j$ in $P$ implies $i < j$.

A consecutive pair $(x_i, x_{i+1})$ is called a jump(or setup) of $P$ in $L$ if $x_i$ is not comparable to $x_{i+1}$. If $x_i < x_{i+1}$ in $P$, then $(x_i, x_{i+1})$ is called a stair(or bump) of $P$ in $L$. Let $s(L,P)[b(L,P)]$ be the number of jumps[stairs] of $P$ in $L$, and let $s(P)[b(P)]$ be the minimum[maximum] of $s(L,P)[b(L,P)]$ over all linear extensions $L$ in $P$. The number $s(P)[b(P)]$ is called the jump[stair] number of

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$P$.

Let $A = [a_{ij}]$ be an $m \times n$ (0,1)-matrix. Let $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_n\}$ be disjoint sets of $m$ and $n$ elements, respectively, and define the order as $x_i < x_j$ iff $a_{ij} = 1$. Then the set $P_A = \{x_1, \ldots, x_m, y_1, \ldots, y_n\}$ with the defined order becomes a poset. For simplicity, $s(A)[b(A)]$ is used for the jump [stair] number of $P_A$.

Let $\Lambda(n,k)$ denote the set of all $(0,1)$-matrices of order $n$ with $k$ 1's in each row and column and $M(n,k) = max\{s(A) : A \in \Lambda(n,k)\}$. In [1], Brualdi and Jung first studied the maximum jump number $M(n,k)$ and gave out its values when $1 \le k \le n \le 10$. They also put forward several conjectures, including the two conjectures that $M(2k+1, k+1) = 4k - \lceil \sqrt{k} \rceil$ for $k \ge 1$ and that $M(n,k) < M(n+l_1, k+l_2)$ for $l_1 \ge 0$, $l_2 \ge 1$, $k \ge 1$. In [2], B. Cheng and B. L. Liu pointed out that the later conjecture does not hold for $l_1 = 0$, $l_2 = 1$. In this paper, we show that $M(2k+1, k+1) = 4k - \lceil \sqrt{k} \rceil$ holds for $5 \le k \le 9$ and that $M(n,k) < M(n+l_1, k+l_2)$ does not hold for $l_1 = 1$, $l_2 = 1$.

Let $J_{a,b}$ denote the $a \times b$ matrix with all 1's, and let $J$ denote any matrix with all 1's of an appropriate size.

The following lemmas obviously hold or come from [1] and [2].

**1 Lemma.** *Let $A$ and $B$ be two $m \times n$(0,1)-matrices. Then*

*(a)* $s(A) + b(A) = m + n - 1$;

*(b)* $s(A \oplus B) = s(A) + s(B) + 1$;

*(c) If there exist two permutation matrices $R$ and $S$ such that $B = RAS$, that is, $A$ can be permuted to $B$, expressed $A \sim B$, then*

*(i)* $b(A) = b(B)$ *and* $s(A) = s(B)$.

*(ii) $A$ and $B$ have the same row sum and column sum.*

**2 Lemma.** $b(A) \ge b(B)$ *holds for every submatrix $B$ of $A$.*

**3 Lemma.** *Let $A$ be a $(0,1)$-matrix with no zero row or column. Let $b(A) = p$. Then there exist permutation matrices $R$ and $S$ and integers $m_1, \ldots, m_p$ and $n_1, \ldots, n_p$ such that $RAS$ equals*

$$\begin{bmatrix} J_{m_1,n_1} & A_{1,2} & \cdots & A_{1,p} \\ O & J_{m_2,n_2} & \cdots & A_{2,p} \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \cdots & J_{m_p,n_p} \end{bmatrix}.$$

**4 Lemma.** $M(2k+1, k+1) \ge 4k - \lceil \sqrt{k} \rceil$ *holds for every positive integer $k$.*

**5 Lemma.** *Let $A$ be a $(0,1)$-matrix with stair number $b(A) = 1$. Then $A$ can be permuted to*

$$J \text{ or } \begin{bmatrix} J & O \end{bmatrix} \text{ or } \begin{bmatrix} J \\ O \end{bmatrix} \text{ or } \begin{bmatrix} J & O \\ O & O \end{bmatrix}.$$

**6 Lemma.** *Let $A$ be a (0,1)-matrix having no rows or columns consisting of all 0's or all 1's. Then $b(A)=2$ if and only if the rows and columns of $A$ can be permuted to an oblique direct sum*

$$O\overline{\oplus}\cdots\overline{\oplus}O$$

*of zero matrices.*

**7 Lemma.** *Let $n$ and $k$ be integers, and let $n\equiv m \pmod{k}$. If $k\mid n$ or $m\mid k$, then $M(n,k)=2n-1-\lceil\frac{n}{k}\rceil$.*

**8 Lemma.** *If $A$ is an $m\times n$ (0,1) matrix without zero row[column] and there are at most $l$ 1's in each column[row], then $b(A)\geq\lceil\frac{m}{l}\rceil[b(A)\geq\lceil\frac{n}{l}\rceil]$.*

# 1 Main Theorem

For a matrix $M$ in block form, we use $M[i_1,i_2,\ldots,i_s|j_1,j_2,\ldots,j_t]$ to denote the submatrix composed of the $i_1$th, $i_2$th, …, $i_s$th block-rows and the $j_1$th, $j_2$th,…, $j_t$th block-columns from $M$. Obviously,

$$b(M)\geq b(M[i_1,i_2,\ldots,i_s|j_1,j_2,\ldots,j_t]).$$

**9 Theorem.** *If $k\geq 6$, then $b(A)\geq 4$ holds for every $A\in\Lambda(2k-1,k)$.*

Proof. Suppose that there exists a matrix $A\in\Lambda(2k-1,k)$ such that $b(A)=3$. Then, according to Lemma 3, we may assume $A$ has the following block triangular form

$$\begin{bmatrix} J_{k,k-q-1} & B_{12} & B_{13} \\ O & J_{p,q} & B_{23} \\ O & O & J_{k-p-1,k} \end{bmatrix},$$

where $1\leq p,\ q\leq k-2$.

Since $b(A)=b(A^T)=3$, we may assume $p\leq q$, $1\leq b(B_{12})\leq b(B_{23})\leq 2$ and $0\leq b(B_{13})\leq 3$. First of all, we have the following lemmas.

**10 Lemma.** *$b(B_{12})=2$.*

Proof. Suppose $b(B_{12})=1$. Since $B_{12}$ has evidently no zero or all 1's column, by Lemma 5 we have $B_{12}\sim\begin{bmatrix} J_{k-p,q} \\ O \end{bmatrix}$, and hence

$$A\sim A_1=\begin{bmatrix} J_{k-p,k-q-1} & J_{k-p,q} & B_1 \\ J_{p,k-q-1} & O & B_2 \\ O & J_{p,q} & B_{23} \\ O & O & J_{k-p-1,k} \end{bmatrix}.$$

The proof will be complete by the following Proposition 11 and Proposition 12 and Proposition 13. $\boxed{QED}$

**11 Proposition.** *$B_1$ has zero columns and $b(B_1) \neq 3$.*

Proof. If $B_1$ has no zero column, then $B_1$ has at least $k$ 1's. On the other hand, each row of $B_1$ has just one 1 since the row sum of $A_1$ equals $k$, and hence $B_1$ has just $k-p$ 1's. It follows $k-p \geq k$, impossible.

If $b(B_1) = 3$, then $b(\begin{bmatrix} B_1 \\ J_{k-p-1,k} \end{bmatrix}) = 4$ since $B_1$ has zero column. Hence $b(A_1) \geq 4$ by Lemma 2, a contradiction. $\boxed{QED}$

**12 Proposition.** *$b(B_1) \neq 1$.*

Proof. If $b(B_1) = 1$, then by Lemma 5 $B_1 \sim \begin{bmatrix} J_{k-p,1} & O \end{bmatrix}$, and hence

$$A_1 \sim A_2 = \begin{bmatrix} J_{k-p,k-q-1} & J_{k-p,q} & J_{k-p,1} & O \\ J_{p,k-q-1} & O & C_1 & C_2 \\ O & J_{p,q} & C_3 & C_4 \\ O & O & J_{k-p-1,1} & J_{k-p-1,k-1} \end{bmatrix}.$$

Obviously, $p \geq \lceil \frac{k-1}{2} \rceil$ and $\begin{bmatrix} C_2 \\ C_4 \end{bmatrix}$ has no zero row or column. It is also clear that $\begin{bmatrix} C_2 \\ C_4 \end{bmatrix}$ has no all 1's column, and hence $b(\begin{bmatrix} C_2 \\ C_4 \end{bmatrix}) = 2$.

If $C_4$ has all 1's rows, then $k \geq q + k - 1$, that is, $0 \geq q - 1 \geq p - 1 \geq \lceil \frac{k-1}{2} \rceil - 1 > 1$ for $k \geq 6$, a contradiction. Hence $b(C_4) = 2$, and by Lemma 3 we have $C_4 \sim \begin{bmatrix} J_{s,t} & * \\ O & J_{p-s,k-t-1} \end{bmatrix}$, where $t = q-1$ or $q$.

If $C_2$ has all 1's rows, then $k-1 \leq (k-q-1)+(k-1) \leq k$, that is, $k = q+1$ or $q+2$, and hence $k = q+2$ due to $k-q-1 \geq 1$. Hence $k \geq q+t = 2q-1$(or $2q$) $= 2(k-2)-1$ (or $2(k-2)$), that is, $k \leq 5$ (or $k \leq 4$), which contradicts $k \geq 6$.

Thus by Lemma 6 we have

$$\begin{bmatrix} C_2 \\ C_4 \end{bmatrix} \sim O_{r,t_1} \overline{\oplus} \cdots \overline{\oplus} O_{r,t_m},$$

where $(m-1)r = p+1$, $mr = 2p$ and $t_1 + \cdots + t_m = k-1$. Hence $p = 3$, $r = 2$ and $m = 3$, and so

$$\begin{bmatrix} C_2 \\ C_4 \end{bmatrix} \sim O_{2,t_1} \overline{\oplus} O_{2,t_2} \overline{\oplus} O_{2,t_3}, t_1 + t_2 + t_3 = k - 1.$$

Since both $C_2$ and $C_4$ have no zero column, we have $b(C_2) = b(C_4) = 2$. Due to $p = 3$ and $p \geq \lceil \frac{k-1}{2} \rceil$, we have $k \leq 7$.

If $k=6$ or 7, then $C_1=O$ or $C_3=O$ or $\begin{bmatrix} C_1 \\ C_3 \end{bmatrix}=O$, and hence

$$b(A_2[1,2,3,4|2,3,4])=4,$$

a contradiction.

Therefore, $b(B_1)\neq 1$. $\boxed{QED}$

**13 Proposition.** *$b(B_1)\neq 2$.*

PROOF. Assume $b(B_1)=2$. Because $B_1$ has no all 1's row or all 1's column, we may suppose $B_1 \sim \begin{bmatrix} J_{s,1}\oplus J_{t,1} & O_{k-p,k-2} \end{bmatrix}$, where $s+t=k-p$. Thus

$$A_1 \sim A_3 = \begin{bmatrix} J_{k-p,k-q-1} & J_{k-p,q} & J_{s,1}\oplus J_{t,1} & O \\ J_{p,k-q-1} & O & D_1 & D_2 \\ O & J_{p,q} & D_3 & D_4 \\ O & O & J_{k-p-1,2} & J_{k-p-1,k-2} \end{bmatrix}.$$

Obviously $b(\begin{bmatrix} D_2 \\ D_4 \end{bmatrix})=1$, and both $D_2$ and $D_4$ have no zero column. It is also clear that $2(k-p-1)+(s+t)\leq 2k$, that is, $k\leq 3p+2$.

If $D_2$ has zero rows, then $k\leq (k-q-1)+2=k-q+1$, that is, $q\leq 1$, which implies $k\geq 5$, contradicting $k\geq 6$.

If $D_4$ has zero rows, then $k\leq q+2$, and hence $q=k-2$ due to $q\leq k-2$. Since we have $D_4\sim \begin{bmatrix} J \\ O \end{bmatrix}$, it follows that $k\geq q+(k-2)=2(k-2)>k$ for $k\geq 6$, a contradiction.

Hence $\begin{bmatrix} D_2 \\ D_4 \end{bmatrix}=J_{2p,k-2}$, and so $2k\geq (k-q-1)+q+2(k-2)$. But $(k-q-1)+q+2(k-2)=3k-5>2k$ for $k\geq 6$, a contradiction. Therefore Proposition 13 holds. $\boxed{QED}$

Due to Lemma 10, we have

**14 Lemma.** *$B_{12}$ has no zero row or zero column.*

**15 Lemma.** *$B_{12}$ has no all 1's column or all 1's row.*

PROOF. $B_{12}$ has obviously no all 1's column.

Suppose $B_{12}$ has $t$ all 1's rows, then $B_{12}\sim \begin{bmatrix} J_{t,q} \\ E \end{bmatrix}$, where

$$E=O_{p,q_1}\overline{\oplus}\cdots\overline{\oplus}O_{p,q_m},\, mp=k-t, q_1+\cdots+q_m=q, \quad m\geq 2,$$

and hence

$$A\sim A_5=\begin{bmatrix} J_{t,k-q-1} & J_{t,q} & E_1 \\ J_{k-t,k-q-1} & E & E_2 \\ O & J_{p,q} & B_{23} \\ O & O & J_{k-p-1,k} \end{bmatrix}.$$

Obviously $E_1$ has zero columns and $1 \leq b(E_1) \leq 2$.

The proof will be complete by the following Proposition 16 and Proposition 17 $\boxed{QED}$

**16 Proposition.** $b(E_1) \neq 2$.

Proof. If $b(E_1) = 2$, then $E_1 \sim \begin{bmatrix} J_{t_1,1} & O & O \\ O & J_{t-t_1,1} & O \end{bmatrix}$, and hence

$$A_4 \sim A_5 = \begin{bmatrix} J_{t_1,k-q-1} & J_{t_1,q} & J_{t_1,1} & O & O \\ J_{t-t_1,k-q-1} & J_{t-t_1,q} & O & J_{t-t_1,1} & O \\ J_{k-t,k-q-1} & E & E_2^{'} & E_2^{''} & E_2^{'''} \\ O & J_{p,q} & B_{23}^{'} & B_{23}^{''} & B_{23}^{'''} \\ O & O & J_{k-p-1,1} & J_{k-p-1,1} & J_{k-p-1,k-2} \end{bmatrix}.$$

Obviously $E_2^{'''}$ has no zero column and $b(E_2^{'''}) = b(B_{23}^{'''}) = 1$.

If $E_2^{'''}$ or $B_{23}^{'''}$ has a submatrix of the form $\begin{bmatrix} J & O \end{bmatrix}$, then $b(A_5) \geq 4$, a contradiction. Hence both $E_2^{'''}$ and $B_{23}^{'''}$ have all 1's rows. It follows that $2k \geq ((k-q-1)+1+(k-2))+(q+k-2) = 3k-4 > 2k$ for $k \geq 6$, a contradiction. Hence $b(E_1) \neq 2$. $\boxed{QED}$

**17 Proposition.** $b(E_1) \neq 1$.

Proof. If $b(E_1) = 1$, then $E_1 \sim \begin{bmatrix} J_{t,1} & O \end{bmatrix}$, and hence

$$A_4 \sim A_6 = \begin{bmatrix} J_{t,k-q-1} & J_{t,q} & J_{t,1} & O \\ J_{k-t,k-q-1} & E & F_3 & F_1 \\ O & J_{p,q} & F_4 & F_2 \\ O & O & J_{k-p-1,1} & J_{k-p-1,k-1} \end{bmatrix}.$$

Obviously $1 \leq b(F_i) \leq 2 (i = 1, 2)$.

If $F_1$ has all 1's columns, then $k \geq (k-t)+(k-p-1) = mp+k-p-1 = k+(m-1)p-1 > k$, a contradiction. Besides, due to $E = O_{p,q_1}\overline{\oplus}\cdots\overline{\oplus}O_{p,q_m} (m \geq 2)$, $F_1$ has no zero row or all 1's row, and hence by Lemma 6 $F_1 \sim O\overline{\oplus}\cdots\overline{\oplus}O$.

The proof will be complete by the following Claim 18 and Claim 19. $\boxed{QED}$

**18 Claim.** $b(F_2) \neq 1$.

Proof. It is clear that $F_2$ has no zero row or all 1's row. If $b(F_2) = 1$, then we may assume $F_2 \sim \begin{bmatrix} J_{p,s} & O \end{bmatrix}$, and hence

$$A_6 \sim A_7 = \begin{bmatrix} J_{t,k-q-1} & J_{t,q} & J_{t,1} & O & O \\ J_{k-t,k-q-1} & E & F_3 & F_1^{'} & F_1^{''} \\ O & J_{p,q} & F_4 & J_{p,s} & O \\ O & O & J_{k-p-1,1} & J_{k-p-1,s} & J_{k-p-1,k-s-1} \end{bmatrix}.$$

Obviously, $F_3 \neq J_{k-t,1}$ and $F_1'' \sim J_{p+1,k-s-1}$ or $\begin{bmatrix} J_{p+1,k-s-1} \\ O \end{bmatrix}$. If $F_1'' \sim J_{p+1,k-s-1}$ , then $k-t = p+1$, and hence $mp = p+1$ or $(m-1)p = 1$, impossible. Hence $F_1'' \sim \begin{bmatrix} J_{p+1,k-s-1} \\ O \end{bmatrix}$. Since each column of $F_1'$ has only one 1, we have $F_1' \sim \begin{bmatrix} J_{1,s} \\ O \end{bmatrix}$ or $\begin{bmatrix} J_{1,s_1} & O \\ O & J_{1,s-s_1} \\ O & O \end{bmatrix}$. But if $F_1' \sim \begin{bmatrix} J_{1,s_1} & O \\ O & J_{1,s-s_1} \\ O & O \end{bmatrix}$, then $b(A_7[1,2,4|1,4,5]) = 4$, a contradiction. Hence $F_1' \sim \begin{bmatrix} J_{1,s} \\ O \end{bmatrix}$.

Since $\begin{bmatrix} F_1' & F_1'' \end{bmatrix}$ has obviously no zero rows, we conclude that $\begin{bmatrix} F_1' & F_1'' \end{bmatrix} \sim \begin{bmatrix} J_{1,s} & O \\ O & J_{p+1,k-s-1} \end{bmatrix}$, and hence $k-t = p+2$, which implies $m = p = 2$ and $k = 4+t$. Due to $k \geq 6$ and the column sum of $A_7$ equals $k$, we have $F_4 = O$ and $t = 2$ or 3. If $t = 3$, then $F_3 = O$, and hence $b(A_7[1,2|2,3,4]) = 4$, a contradiction. Hence $t = 2$ and $F_3 \sim \begin{bmatrix} J_{1,1} \\ O \end{bmatrix}$. It follows that $\begin{bmatrix} F_3 & F_1' & F_1'' \end{bmatrix} \sim \begin{bmatrix} J_{1,1} & J_{1,s} & O \\ O & O & J_{3,5-s} \end{bmatrix}$ or $\begin{bmatrix} O & J_{1,s} & O \\ J_{1,1} & O & J_{1,5-s} \\ O & O & J_{2,5-s} \end{bmatrix}$, which implies $b(A_7[1,2|2,3,4,5]) = 4$ or $b(A_7[2,3|1,3,4,5]) = 4$, a contradiction.

Therefore Claim 18 holds. $\boxed{QED}$

**19 Claim.** $b(F_2) \neq 2$.

Proof. Assume $b(F_2) = 2$. Then $F_2$ has no zero column. Let

$$F_2 \sim \begin{bmatrix} J_{p,r} & O\overline{\oplus}\cdots\overline{\oplus}O \end{bmatrix} \ (r \geq 0).$$

If $r > 0$, then

$$F_1 \sim \begin{bmatrix} J_{1,r} & O \\ O & J_{k-t-1,k-r-1} \end{bmatrix},$$

and hence $k \geq (k-t-1)+1+(k-p-1) = k+(m-1)p-1 > k$, a contradiction. Hence $r = 0$, and so $F_2 \sim O_{p_1,b_1}\overline{\oplus}\cdots\overline{\oplus}O_{p_h,b_h}$, where $p_1+\cdots+p_h = p$, $b_1+\cdots+b_h = k-1$.

**(a).** If $F_4 = J_{p,1}$, then $b_1 = \cdots = b_h = q$, $hq = k-1$, $F_3 = O$ and $t = 1$.

Since $F_1$ is a $(k-1)\times(k-1)$ (0,1)-matrix without zero row or column, and there are at most $p$ 1's in each column and at most $q$ 1's in each row, and hence by Lemma 8 we have $b(F_1) \geq \lceil \frac{k-1}{p} \rceil = \lceil \frac{mp}{p} \rceil = m$ and $b(F_1) \geq \lceil \frac{k-1}{q} \rceil = \lceil \frac{hq}{q} \rceil = h$.

If $m \geq 3$ or $h \geq 3$, then $b(F_1) \geq 3$, and hence

$$b(A_6) \geq b(\begin{bmatrix} J_{t,k-q-1} & O \\ J_{k-t,k-q-1} & F_1 \end{bmatrix}) \geq 4,$$

a contradiction. Hence $m = h = 2$, which implies $k = 2p + 1$ and $p = q$, and it follows that

$$A_6 \sim A_8 = \begin{bmatrix} J_{1,k-q-1} & J_{1,q_1} & J_{1,q-q_1} & J_{1,1} & O & O \\ J_{p,k-q-1} & J_{p,q_1} & O & O & H_1 & H_2 \\ J_{p,k-q-1} & O & J_{p,q-q_1} & O & H_3 & H_4 \\ O & J_{p_1,q_1} & J_{p_1,q-q_1} & J_{p_1,1} & J_{p_1,q} & O \\ O & J_{p-p_1,q_1} & J_{p-p_1,q-q_1} & J_{p-p_1,1} & O & J_{p-p_1,q} \\ O & O & O & J_{k-p-1,1} & J_{k-p-1,q} & J_{k-p-1,q} \end{bmatrix}.$$

Obviously $b(H_i) \leq 1 (i = 1, 2, 3, 4)$.

Without loss of generality, we assume $p_1 \leq p - p_1$ and $q_1 \leq q - q_1$.

Since $\begin{bmatrix} H_1 & H_2 \end{bmatrix}$ is a $p \times 2q$ (0,1)-matrix without zero row and there are at most $p - p_1 + 1$ 1's in each column, by Lemma 8 $b(\begin{bmatrix} H_1 & H_2 \end{bmatrix}) \geq \lceil \frac{p}{p-p_1+1} \rceil \geq 1$ and the equality holds iff $p = p - p_1 + 1$, that is, $p_1 = 1$. If $b(\begin{bmatrix} H_1 & H_2 \end{bmatrix}) > 1$, then $b(A_8) \geq 4$, a contradiction. Hence $p_1 = 1$. Similarly, $q_1 = 1$.

If $H_i \sim \begin{bmatrix} J & O \\ O & O \end{bmatrix}$ or $\begin{bmatrix} J \\ O \end{bmatrix}$ or $\begin{bmatrix} J & O \end{bmatrix}$, then we have $b(A_8) \geq 4$. Hence $H_i \sim O$ or $J$, and so $\begin{bmatrix} H_1 & H_2 \\ H_3 & H_4 \end{bmatrix} \sim \begin{bmatrix} J_{p,q} & O \\ O & J_{p,q} \end{bmatrix}$. Thus, $k = p + (p - p_1) + (k - p - 1)$ or $p = p_1 + 1 = 2$, and hence $k = 2p + 1 = 5$, which contradicts $k \geq 6$.

**(b).** If $F_4 \sim \begin{bmatrix} J_{d,1} \\ O \end{bmatrix}$, then

$$A_6 \sim A_9 = \begin{bmatrix} J_{t,k-q-1} & J_{t,q} & J_{t,1} & O \\ J_{k-t,k-q-1} & E & F_3 & F_1 \\ O & J_{d,q} & J_{d,1} & F_2' \\ O & J_{p-d,q} & O & F_2'' \\ O & O & J_{k-p-1} & J_{k-p-1,k-1} \end{bmatrix}.$$

Obviously $F_2'' \sim \begin{bmatrix} J_{p-d,k-q} & O \end{bmatrix}$, and so

$$A_9 \sim A_{10} = \begin{bmatrix} J_{t,k-q-1} & J_{t,q} & J_{t,1} & O & O \\ J_{k-t,k-q-1} & E & F_3 & G_1 & G_2 \\ O & J_{d,q} & J_{d,1} & G_3 & G_4 \\ O & J_{p-d,q} & O & J_{p-d,k-q} & O \\ O & O & J_{k-p-1,1} & J_{k-p-1,k-q} & J_{k-p-1,q-1} \end{bmatrix}.$$

It is clear that $b(G_4) = 1$. If $G_4 \sim \begin{bmatrix} J & O \end{bmatrix}$ or $\begin{bmatrix} J & O \\ O & O \end{bmatrix}$, then $b(A_{10}[1, 3, 4, 5|3, 4, 5]) = 4$, a contradiction. If $G_4 \sim \begin{bmatrix} J \\ O \end{bmatrix}$, then $b(A_{10}[1, 3, 4|1, 2, 3, 5]) = 4$, a contradiction. If $G_4 = O$, then $G_3 \sim \begin{bmatrix} J_{d,k-q-1} & O \end{bmatrix}$, and hence $b(A_{10}[1, 3, 4, 5|3, 4, 5]) = 4$, a contradiction. Hence $G_4 = J_{d,q-1}$, and so $G_2 \sim \begin{bmatrix} J_{l,q-1} \\ O \end{bmatrix}$ $(l = p+1-d)$. It follows that

$$A_{10} \sim A_{11} = \begin{bmatrix} J_{t,k-q-1} & J_{t,q} & J_{t,1} & O & O \\ J_{l,k-q-1} & E^{'} & F_3^{'} & G_1^{'} & J_{l,q-1} \\ J_{k-t-l,k-q-1} & E^{''} & F_3^{''} & G_1^{''} & O \\ O & J_{d,q} & J_{d,1} & G_3 & J_{d,g-1} \\ O & J_{p-d,q} & O & J_{p-d,k-q} & O \\ O & O & J_{k-p-1,1} & J_{k-p-1,k-q} & J_{k-p-1,q-1} \end{bmatrix}.$$

First, we have $F_3^{''} \not\sim O$ or $\begin{bmatrix} J \\ O \end{bmatrix}$, otherwise $b(A_{11}[1,3,4,5|1,2,3,5]) = 4$, a contradiction.

Second, we have $F_3^{''} \neq J_{k-t-l,1}$, otherwise $k \geq t+(k-t-l)+d+(k-p-1)$, that is, $0 \geq 2(d-1)+t+(m-2)p$ , impossible.

Hence Claim 19 holds. $\boxed{QED}$

Next, we continue the proof of Theorem 9.

By Lemma 10, Lemma 14 and Lemma 15, we have

$B_{12} \sim O_{p,t_1}\overline{\oplus}\cdots\overline{\oplus}O_{p,t_n}$, $t_1+\cdots+t_n = q$, $k = np$.

Similarly, we have

$B_{23} \sim O_{s_1,q}\overline{\oplus}\cdots\overline{\oplus}O_{s_m,q}$, $s_1+\cdots+s_m = p$, $k = mq$.

Since $B_{13}$ has obviously no zero row or column and no all 1's row or column, and each column(or row) of $B_{13}$ has at most $p$(or $q$) 1's, by Lemma 8 we have $b(B_{13}) \geq \lceil \frac{k}{p} \rceil = \lceil \frac{np}{p} \rceil = n$ and $b(B_{13}) \geq \lceil \frac{k}{q} \rceil = \lceil \frac{mq}{q} \rceil = m$. While $b(B_{13}) = 2$ or 3, and hence $n$=2 or 3 and $m$=2 or 3. Due to the assumption $p \leq q$, we have $(n,m)$=(2,2) or (3,2) or (3,3).

Now we continue our proof in the following three steps.

**(a).** Let $(n,m)$=(2,2). Then $p = q$ and $k = 2p$, and hence

$$A \sim A_{12} = \begin{bmatrix} J_{p,k-p-1} & J_{p,t_1} & O & L_1 & L_2 \\ J_{p,k-p-1} & O & J_{p,p-t_1} & L_3 & L_4 \\ O & J_{s_1,t_1} & J_{s_1,p-t_1} & J_{s_1,p} & O \\ O & J_{p-s_1,t_1} & J_{p-s_1,p-t_1} & O & J_{p-s_1,p} \\ O & O & O & J_{k-p-1,p} & J_{k-p-1,p} \end{bmatrix}.$$

Without loss of generality, we assume $O \leq b(L_1) \leq b(L_2) \leq 2$.

Let $L_1 = O$, then $b(L_2) = 1$, and hence $p - s_1 = 1$ and $L_2 \sim J_{p,p}$ or $\begin{bmatrix} J_{p,p+1-t_1} & O \end{bmatrix}$. It follows that $L_4$ has zero columns. Hence $L_3$ is a submatrix of stair number $b(L_3) = 1$ and has no zero column or zero row, which implies $L_3 = J_{p,p}$, and hence $s_1 = 1$. Thus, $k = 2p = 2(s_1+1) = 4$, contradicting $k \geq 6$.

Let $b(L_1) = 1$. If $L_1 \sim \begin{bmatrix} J & O \\ O & O \end{bmatrix}$, then $b(A_{13}[1,3,4|1,3,4]) = 4$, a contradiction. If $L_1 = J$, then $L_4 = J$, and hence $t_1 = p - t_1 = 1$, which implies $p = 2$, and so $k = 4$, contradicting $k \geq 6$. Similarly, we will also have a contradiction if $L_1 \sim \begin{bmatrix} J & O \end{bmatrix}$ of $\begin{bmatrix} J \\ O \end{bmatrix}$.

Let $b(L_1) = b(L_2) = 2$. Then both $L_1$ and $L_2$ have no zero row or zero column, and hence $\begin{bmatrix} L_1 & L_2 \end{bmatrix}$ is a $p \times 2p$ (0,1) matrix without zero column, and there are at most $p + 1 - t_1$ 1's in its each row. By Lemma 8 we conclude that

$$b(\begin{bmatrix} L_1 & L_2 \end{bmatrix}) \geq \lceil \frac{2p}{p+1-t_1} \rceil = \lceil \frac{2p}{p-(t_1-1)} \rceil = 2 + \lceil \frac{2(t_1-1)}{p-(t_1-1)} \rceil \geq 2,$$

where the equality holds if and only if $t_1 = 1$.

If $t_1 > 1$, then $b(\begin{bmatrix} L_1 & L_2 \end{bmatrix}) > 2$, and hence $b(A_{12}[1,3|3,4,5]) \geq 4$, a contradiction. Thus, $t_1 = 1$. Similarly, we have $p - t_1 = 1$. It follows that $p = 2$ and $k = 2p = 4$, which contradicts $k \geq 6$.

**(b).** Let $(n,m) = (3,2)$. Then we have $k = 3p = 2q$ and hence

$$A \sim A_{13} = \begin{bmatrix} J_{p,k-q-1} & J_{p,t_1} & J_{p,t_2} & O & K_1 & K_2 \\ J_{p,k-q-1} & J_{p,t_1} & O & J_{p,q-t_1-t_2} & K_3 & K_4 \\ J_{p,k-q-1} & O & J_{p,t_2} & J_{p,q-t_1-t_2} & K_5 & K_6 \\ O & J_{s_1,t_1} & J_{s_1,t_2} & J_{s_1,q-t_1-t_2} & J_{s_1,q} & O \\ O & J_{p-s_1,t_1} & J_{p-s_1,t_2} & J_{p-s_1,q-t_1-t_2} & O & J_{p-s_1,q} \\ O & O & O & O & J_{k-p-1,q} & J_{k-p-1,q} \end{bmatrix}.$$

Without loss of generality we assume $0 \leq b(K_1) \leq b(K_2) \leq 2$.

Let $K_1 = O$. Then $b(K_2) = 1$, and hence $K_2 \sim J_{p,q}$ or $\begin{bmatrix} J_{p,l} & O \end{bmatrix}$ ($l = q+1-t_1-t_2$). If $K_2 \sim J_{p,q}$, then $t_1 + t_2 = 1$, impossible. If $K_2 \sim \begin{bmatrix} J_{p,l} & O \end{bmatrix}$, then $p - s_1 = 1$ and $\begin{bmatrix} K_2 \\ K_4 \\ K_6 \end{bmatrix} \sim \begin{bmatrix} J_{p,l} & O \\ O & K_4' \\ O & K_6' \end{bmatrix}$, where $b(\begin{bmatrix} K_4' \\ K_6' \end{bmatrix}) = 1$, and each column of $\begin{bmatrix} K_4' \\ K_6' \end{bmatrix}$ has just $p$ 1's. Hence $\begin{bmatrix} K_4' \\ K_6' \end{bmatrix} \sim \begin{bmatrix} J_{p,q-l} \\ O \end{bmatrix}$, and it follows $b(A_{13}[2,3,6|1,2,6]) = 4$, a contradiction.

Let $b(K_1) = 1$. Due to $t_1 + t_2 \geq 2$, we have that both $K_1$ and $K_2$ have no all 1's row, and hence $K_1 \sim \begin{bmatrix} J_{p,t} & O \end{bmatrix}$ ($1 \leq t < q$). Thus $K_2 \sim \begin{bmatrix} J_{p,q+1-t_1-t_2-t} & O \end{bmatrix}$, and so $s_1 = p - s_1 = 1$, which implies $p = 2$ and $k = 3p = 6$. Hence $t_1 = t_2 = t = 1$ and $\begin{bmatrix} K_3 \\ K_5 \end{bmatrix} \sim \begin{bmatrix} O & J_{2,2} \\ O & O \end{bmatrix}$ or $\begin{bmatrix} O & J_{2,1} & O \\ O & O & J_{2,1} \end{bmatrix}$, and it follows $b(A_{13}[2,3,4|2,3,5]) = 4$, a contradiction.

Let $b(K_1) = b(K_2) = 2$. Then both $K_1$ and $K_2$ have no zero row or zero column, and hence $\begin{bmatrix} K_1 & K_2 \end{bmatrix}$ is a $p \times 2q$ (0,1)-matrix without zero column and its each row has at most $q+1-t_1-t_2$ 1's. Thus by Lemma 8 we have $b(\begin{bmatrix}K_1 & K_2\end{bmatrix}) \geq \lceil \frac{2q}{q+1-t_1-t_2} \rceil = 2 + \lceil \frac{2(t_1+t_2-1)}{q+1-t_1-t_2} \rceil = 3$, and so $b(A_{13}[1,2|4,5,6]) = 4$, a contradiction.

**(c).** Let $(m,n) = (2,3)$. Then we have $k = 3p = 3q$ and $p = q$, and hence $A \sim A_{14} =$

$$\begin{bmatrix}
J_{p,k-p-1} & J_{p,t_1} & J_{p,t_2} & O & N_1 & N_2 & N_3 \\
J_{p,k-p-1} & J_{p,t_1} & O & J_{p,p-t_1-t_2} & * & * & * \\
J_{p,k-p-1} & O & J_{p,t_2} & J_{p,p-t_1-t_2} & * & * & * \\
O & J_{s_1,t_1} & J_{s_1,T_2} & J_{s_1,p-t_1-t_2} & J_{s_1,p} & J_{s_1,p} & O \\
O & J_{s_2,t_1} & J_{s_2,t_2} & J_{s_2,p-t_1-t_2} & J_{s_2,p} & O & J_{s_2,p} \\
O & J_{p-s_1-s_2,t_1} & J_{p-s_1-s_2,t_2} & J_{p-s_1-s_2,p-t_1-t_2} & O & J_{p-s_1-s_2,p} & J_{p-s_1-s_2,p} \\
O & O & O & O & J_{k-p-1,p} & J_{k-p-1,p} & J_{k-p-1,p}
\end{bmatrix}$$

where $*$ denotes any matrix of appropriate size.

Without loss of generality we assume $0 \leq b(N_1) \leq b(N_2) \leq b(N_3) \leq 2$.

Let $N_1 = O$, then $b(\begin{bmatrix}N_2 & N_3\end{bmatrix}) = 1$ and $\begin{bmatrix}N_2 & N_3\end{bmatrix}$ has no zero row, and hence $\begin{bmatrix}N_2 & N_3\end{bmatrix} \sim \begin{bmatrix}J & O\end{bmatrix}$. It follows $p - s_1 - s_2 \leq 0$, impossible.

If $b(N_2) = 1$, then $\begin{bmatrix}N_2 & N_3\end{bmatrix} \sim \begin{bmatrix}J_{p,t} & O & J_{p,l} & O\end{bmatrix}$ $(t+l = p+1-t_1-t_2)$. Thus we also have $p - s_1 - s_2 \leq 0$, impossible.

Let $b(N_1) = 1$. If $N_1 \sim J$ or $\begin{bmatrix}J \\ O\end{bmatrix}$, then $t_1 + t_2 \leq 1$, impossible. If $N_1 \sim \begin{bmatrix}J & O \\ O & O\end{bmatrix}$, then $b(A_{14}[1,6,7|2,4,5]) = 4$, a contradiction. If $N_1 \sim \begin{bmatrix}J & O\end{bmatrix}$, then we have $k \geq p + s_1 + s_2 + (k-p-1)$, that is, $1 \geq s_1 + s_2$, impossible.

Let $b(N_1) = 2$, then $b(N_2) = b(N_3) = 2$, and hence $\begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}$ is a $p \times 3p$ (0,1)-matrix without zero column, and there are at most $p+1-t_1-t_2$ 1's in its each row. Thus, by Lemma 8 we have $b(\begin{bmatrix}N_1 & N_2 & N_3\end{bmatrix}) \geq \lceil \frac{3p}{p+1-t_1-t_2} \rceil = 3 + \lceil \frac{3(t_1+t_2-1)}{p+1-t_1-t_2} \rceil = 4$, which implies $b(A_{14}) \geq 4$, a contradiction.

By the above showed, we have proved that there does not exist a $A \in \Lambda(2k-1,k)$ such that $b(A) = 3$ for $k \geq 6$, which implies Theorem 9 holds. $\boxed{QED}$

## 2 Corollaries

**20 Corollary.** $4(k-1)-\lceil\sqrt{k-1}\rceil \leq M(2k-1,k) \leq 4k-7$ *holds for* $k \geq 6$.

Proof. By Lemma 4, we have $M(2k-1,k) \geq 4(k-1)-\lceil\sqrt{k-1}\rceil$. On the other hand, by Theorem 9 $M(2k-1,k) \leq 2(2k-1)-1-4 = 4k-7$. Hence Corollary 20 holds. $\boxed{QED}$

**21 Corollary.** *Brualdi's conjecture* $M(2k+1,k+1) = 4k-\lceil\sqrt{k}\rceil$ *holds for* $k$*=5, 6, 7, 8 and 9.*

Proof. Trivial by Corollary 20. $\boxed{QED}$

**22 Corollary.** *Brualdi's conjecture* $M(n,k) < M(n+l_1,k+l_2)$ *does not hold for* $l_1 = 1$, $l_2 = 1$.

Proof. By Lemma 7 $M(2k,k) = 4k-1-\lceil\frac{2k}{k}\rceil = 4k-3$. While by Corollary 21 $M(2k+1,k+1) = 4k-3$ holds for $k$=5, 6, 7, 8 and 9. Hence $M(2k,k) = M(2k+1,k+1)$ holds for $k$=5, 6, 7, 8 and 9. Therefore $M(n,k) < M(n+l_1,k+l_2)$ does not hold for $n = 2k$ and $l_1 = l_2 = 1$ and $k$=5, 6, 7, 8 and 9. $\boxed{QED}$

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